

ON THE BV QUANTIZATION OF GAUGE GRAVITATION THEORY

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Quantization of gravitation theory as gauge theory of general covariant transformations in the framework of Batalin–Vilkoviski (BV) formalism is considered. Its gauge-fixed Lagrangian is constructed.

1 Introduction

Classical gauge gravitation theory has been developed in different variants since 60th (see [1, 2, 3] for a survey). We describe gravitation theory as a standard gauge theory on fiber bundles where parameters of gauge transformations are vector fields on a base manifold X . The well-known BV quantization technique [4, 5] can be applied to this theory. Since this quantization technique fails to provide a functional measure, we restrict ourselves to constructing a gauge-fixed Lagrangian.

Recall that an r -order Lagrangian of a Lagrangian system on a smooth fiber bundle $Y \rightarrow X$ is defined as a density

$$L = \mathcal{L}\omega : J^r Y \rightarrow \bigwedge^n T^* X, \quad \omega = dx^1 \wedge \cdots \wedge dx^n, \quad n = \dim X, \quad (1)$$

on the r -order jet manifold $J^r Y$ of sections of $Y \rightarrow X$ [6, 7]. Given bundle coordinates (x^λ, y^i) on Y , the jet manifold $J^r Y$ is endowed with the adapted coordinates $(x^\lambda, y^i, y_\Lambda^i)$, where $\Lambda = (\lambda_k \dots \lambda_1)$, $k = 1, \dots, r$, is a symmetric multi-index. We use the notation $\lambda + \Lambda = (\lambda \lambda_k \dots \lambda_1)$ and

$$d_\lambda = \partial_\lambda + \sum_{0 \leq |\Lambda|} y_{\lambda+\Lambda}^i \partial_i^\Lambda, \quad d_\Lambda = d_{\lambda_r} \circ \cdots \circ d_{\lambda_1}. \quad (2)$$

A Lagrangian system on a fiber bundle is said to be a gauge system if its Lagrangian L admits a gauge symmetry depending on parameter functions and their derivatives. In order to describe a gauge system, let us consider Lagrangian formalism on the bundle product

$$E = Y \times_X W, \quad (3)$$

where $W \rightarrow X$ is a vector bundle whose sections are gauge parameter functions [8]. Let $W \rightarrow X$ be coordinated by (x^λ, ξ^r) . Then a gauge symmetry is represented by a differential operator

$$v = \sum_{0 \leq |\Lambda| \leq m} v_r^{i,\Lambda}(x^\lambda, y_\Sigma^i) \xi_\Lambda^r \partial_i \quad (4)$$

on E (3) which is linear on W and takes its values in the vertical tangent bundle VY of $Y \rightarrow X$. By means of a replacement of even gauge parameters ξ^r and their jets ξ_Λ^r with the odd ghosts c^r and their jets c_Λ^r , the operator (4) defines a graded derivation

$$v = \sum_{0 \leq |\Lambda| \leq m} v_r^{i,\Lambda}(x^\lambda, y_\Sigma^i) c_\Lambda^r \partial_i \quad (5)$$

of the algebra of the original even fields and odd ghosts. Its extension

$$v = \sum_{0 \leq |\Lambda| \leq m} v_r^{i,\Lambda} c_\Lambda^r \partial_i + u^r \partial_r \quad (6)$$

to ghosts is called the BRST transformation if it is nilpotent. Such an extension exists if the original gauge transformations form an algebra [9].

As was mentioned above, gauge transformations in gauge gravitation theory are parameterized by vector fields on X . In this case, the vector bundle $W \rightarrow X$ possesses the composite fibration $W \rightarrow TX \rightarrow X$, where TX is the tangent bundle of X . Here, we are concerned with the following three gauge gravitation theories: (i) the metric-affine gravitation theory on natural bundles describing gravity in the absence of matter fields, (ii) gauge theory on principal bundles with non-vertical gauge transformations which describes gauge potentials of internal symmetries in the presence of a gravitational field, (iii) gauge gravitation theory in the presence of Dirac fermion fields. The corresponding gauge transformations (12), (16), (38), (39), gauge symmetries (59), (61), (63), (65), and BRST symmetries (87), (88), (89) are obtained. However, we apply the BV quantization procedure only to the first of these gauge models describing pure gravity. We construct its extended Lagrangian (98), depending on ghost and antifields, and then the gauge-fixed Lagrangian (104), where antifields are replaced with gauge-fixing terms.

2 Gauge gravitation theories

Classical theory of gravity in the absence of matter fields can be formulated as a gauge theory on natural bundles over an oriented four-dimensional manifold X . The group of automorphisms of a natural bundle contains the subgroup of general covariant transformations defined as the canonical lift of diffeomorphisms of X . They are gauge transformations of metric-affine gravitation theory on natural bundles whose variables are linear connections and pseudo-Riemannian metrics on X .

Linear connections on X (henceforth world connection) are connections on the principal bundle LX of linear frames in the tangent bundle TX of X with the structure group $GL_4 = GL^+(4, \mathbb{R})$. These connections are represented by sections of the quotient bundle

$$C_K = J^1 LX / GL_4, \quad (7)$$

where $J^1 LX$ is the first order jet manifold of sections of $LX \rightarrow X$ [6, 7]. This fiber bundle is provided with bundle coordinates $(x^\lambda, k_\lambda{}^\nu{}_\alpha)$ such that, for any section K of $C_K \rightarrow X$, its coordinates $k_\lambda{}^\nu{}_\alpha \circ K = K_\lambda{}^\nu{}_\alpha$ are coefficient of the linear connection

$$K = dx^\lambda \otimes (\partial_\lambda + K_\lambda{}^\mu{}_\nu \dot{x}^\nu \dot{\partial}_\mu)$$

on TX with respect to the holonomic bundle coordinates $(x^\lambda, \dot{x}^\lambda)$. Note that the first order jet manifold $J^1 C_K$ of the fiber bundle (7) possesses the canonical decomposition taking the coordinate form

$$k_{\lambda\mu}{}^\alpha{}_\beta = \frac{1}{2}(R_{\lambda\mu}{}^\alpha{}_\beta + S_{\lambda\mu}{}^\alpha{}_\beta), \quad (8)$$

$$R_{\lambda\mu}{}^\alpha{}_\beta = k_{\lambda\mu}{}^\alpha{}_\beta - k_{\mu\lambda}{}^\alpha{}_\beta + k_\mu{}^\alpha{}_\varepsilon k_\lambda{}^\varepsilon{}_\beta - k_\lambda{}^\alpha{}_\varepsilon k_\mu{}^\varepsilon{}_\beta, \quad (8)$$

$$S_{\lambda\mu}{}^\alpha{}_\beta = k_{\lambda\mu}{}^\alpha{}_\beta + k_{\mu\lambda}{}^\alpha{}_\beta - k_\mu{}^\alpha{}_\varepsilon k_\lambda{}^\varepsilon{}_\beta + k_\lambda{}^\alpha{}_\varepsilon k_\mu{}^\varepsilon{}_\beta. \quad (9)$$

If K is a section of $C_K \rightarrow X$, then $R \circ K$ is the curvature of the world connection K .

In order to describe gravity, we assume that the linear frame bundle LX admits a Lorentz structure, i.e., a principal subbundle $L^h X \subset LX$ whose structure group is the proper Lorentz group $L = SO^0(1, 3)$. By virtue of the well-known theorem [10], there is one-to-one correspondence between these subbundles and the global sections h of the quotient bundle

$$\Sigma_T = LX / L. \quad (10)$$

Namely, $L^h X = h^* LX$ is the pull-back of $LX \rightarrow \Sigma_T$ by h . We agree to call Σ_T the tetrad bundle. This is the twofold of the metric bundle

$$\Sigma = LX / SO(1, 3), \quad (11)$$

whose sections are pseudo-Riemannian (henceforth world) metrics on X . Being an open subbundle of the tensor bundle $\overset{2}{V} TX$, the bundle Σ (11) is provided with bundle coordinates $(x^\lambda, \sigma^{\mu\nu})$.

The configuration space of gauge gravitation theory in the absence of matter fields is the bundle product $\Sigma \times C_K$. This is a natural bundle admitting the canonical lift

$$u_{K\Sigma} = u^\mu \partial_\mu + (\sigma^{\nu\beta} \partial_\nu u^\alpha + \sigma^{\alpha\nu} \partial_\nu u^\beta) \frac{\partial}{\partial \sigma^{\alpha\beta}} + \quad (12)$$

$$(\partial_\nu u^\alpha k_\mu{}^\nu{}_\beta - \partial_\beta u^\nu k_\mu{}^\alpha{}_\nu - \partial_\mu u^\nu k_\nu{}^\alpha{}_\beta + \partial_{\mu\beta} u^\alpha) \frac{\partial}{\partial k_\mu{}^\alpha{}_\beta}$$

of vector fields $u = u^\lambda \partial_\lambda$ on X [6]. This lift is a Lie algebra homomorphism, and the vector fields (12) are infinitesimal generators of local one-dimensional groups of general covariant transformations, whose gauge parameters are vector fields on X .

Dealing with non-vertical automorphisms of non-natural bundles, one also comes to gravitation theory. For instance, let us consider the gauge theory on a principal bundle $P \rightarrow X$ with a structure Lie group G . In a general setting, the gauge-natural prolongations of P and the associated natural-gauge bundles are called into play [11, 12]. Principal connections on P are represented by sections of the quotient

$$C = J^1 P / G \rightarrow X, \quad (13)$$

called the bundle of principal connections [6]. This is an affine bundle coordinated by (x^λ, a_λ^r) such that, given a section A of $C \rightarrow X$, its components $A_\lambda^r = a_\lambda^r \circ A$ are coefficients of the familiar local connection form (i.e., gauge potentials). Infinitesimal generators of local one-parameter groups of automorphisms of a principal bundle P are G -invariant projectable vector fields on $P \rightarrow X$. They are represented by sections of the vector bundle $T_G P = TP/G \rightarrow X$. This fiber bundle is endowed with the coordinates $(x^\lambda, \tau^\lambda = \dot{x}^\lambda, \xi^r)$ with respect to the fiber bases $\{\partial_\lambda, e_r\}$ for $T_G P$, where $\{e_r\}$ is the basis for the right Lie algebra \mathfrak{g} of G such that $[e_p, e_q] = c_{pq}^r e_r$. The bracket of sections of $T_G P \rightarrow X$ reads

$$\begin{aligned} u_P &= u^\lambda \partial_\lambda + u^r e_r, & v_P &= v^\lambda \partial_\lambda + v^r e_r, \\ [u_P, v_P] &= (u^\mu \partial_\mu v^\lambda - v^\mu \partial_\mu u^\lambda) \partial_\lambda + (u^\lambda \partial_\lambda v^r - v^\lambda \partial_\lambda u^r + c_{pq}^r u^p v^q) e_r. \end{aligned} \quad (14)$$

Any section u_P of the vector bundle $T_G P \rightarrow X$ yields the vector field

$$u_C = u^\lambda \partial_\lambda + (c_{pq}^r a_\lambda^p u^q + \partial_\lambda u^r - a_\mu^r \partial_\lambda u^\mu) \partial_r^\lambda \quad (15)$$

on the bundle of principal connections C (13). For instance, this vector field is a gauge symmetry of the global Chern–Simons Lagrangian in gauge theory on a principal bundle with a structure semi-simple Lie group G over a three-dimensional base X [13]. In order to obtain a gauge symmetry of the Yang–Mills Lagrangian, one have to extend the vector field (15) to the vector field

$$u_{C\Sigma} = u^\lambda \partial_\lambda + (c_{pq}^r a_\lambda^p u^q + \partial_\lambda u^r - a_\mu^r \partial_\lambda u^\mu) \partial_r^\lambda + (\sigma^{\nu\beta} \partial_\nu u^\alpha + \sigma^{\alpha\nu} \partial_\nu u^\beta) \frac{\partial}{\partial \sigma^{\alpha\beta}} \quad (16)$$

on the bundle product $C \times \Sigma$ of C and the metric bundle Σ (11).

The physical underlying reasons for the existence of a Lorentz structure and, consequently, a world metric are Dirac fermion fields. They are described by sections of Dirac spinor bundles, which fail to be natural. In order to develop gauge theory of gravity in the presence of Dirac fermion fields, one has to define general covariant transformations of these fields.

Recall that Dirac spinors are described in terms of Clifford algebras as follows [14, 15]. Let M be the Minkowski space equipped with the Minkowski metric η of signature $(+---)$, and let $\{t^a\}$ be a fixed basis for M . The complex Clifford algebra $\mathbb{C}_{1,3}$ is defined as the complexified quotient of the tensor algebra $\otimes M$ of M by the two-sided ideal generated by elements

$$t \otimes t' + t' \otimes t - 2\eta(t, t') \in \otimes M, \quad t, t' \in M.$$

The Dirac spinor space V_s is introduced as a minimal left ideal of $\mathbb{C}_{1,3}$ on which this algebra acts by left multiplications. There is the representation

$$\gamma : M \otimes V \rightarrow V, \quad \gamma(t^a) = \gamma^a, \quad (17)$$

of elements of the Minkowski space $M \subset \mathbb{C}_{1,3}$ by the Dirac γ -matrices on V_s . Different ideals V_s lead to equivalent representations (17). The complex Clifford algebra $\mathbb{C}_{1,3}$ is isomorphic to the real Clifford algebra $\mathbb{R}_{2,3}$, whose generating space is \mathbb{R}^5 endowed with the metric of signature $(+ - - - +)$. Its subalgebra generated by elements of $M \subset \mathbb{R}^5$ is the real Clifford algebra $\mathbb{R}_{1,3}$. The Clifford group $G_{1,3}$ consists of the invertible elements l_s of $\mathbb{R}_{1,3}$ such that the inner automorphisms defined by these elements preserve the Minkowski space $M \subset \mathbb{R}_{1,3}$. Since this action of $G_{1,3}$ on M is not effective, one consider its spin subgroup $L_s = SL(2, \mathbb{C})$, acting on the spinor space V_s by means of the generators

$$L_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]. \quad (18)$$

This action preserves the representation (17) and the spinor metric

$$a(v, v') = \frac{1}{2}(v^+ \gamma^0 v' + v'^+ \gamma^0 v), \quad v, v' \in V_s. \quad (19)$$

The spin group L_s is the universal twofold of the proper Lorentz group L . Given the universal twofold \widetilde{GL}_4 of the group GL_4 , we have the commutative diagram

$$\begin{array}{ccc} \widetilde{GL}_4 & \longrightarrow & GL_4 \\ \uparrow & & \uparrow \\ L_s & \longrightarrow & L \end{array} \quad (20)$$

Note that the group \widetilde{GL}_4 admits spinor representations, but they are infinite-dimensional [2, 16]. Let us consider the \widetilde{GL} -principal bundle $\widetilde{LX} \rightarrow X$ which is the twofold of the linear frame bundle LX [17, 18, 19]. We agree to call it the spin frame bundle. One says that \widetilde{LX} admits a Dirac spin structure if there exists its L_s -principal subbundle P^h . Then we have the commutative diagram

$$\begin{array}{ccc} \widetilde{LX} & \xrightarrow{\zeta} & \widetilde{LX} \\ \uparrow & & \uparrow \\ P^h & \longrightarrow & L^h X \end{array} \quad (21)$$

where $L^h X$ is some Lorentz structure [20, 21]. As a consequence, $\widetilde{LX}/L_s = \Sigma_T$ (10), and there is one-to-one correspondence $P^h = h^* \widetilde{LX}$ between the Dirac spin structures $P^h \subset \widetilde{LX}$ and the global sections h of the tetrad bundle $\Sigma_T \rightarrow X$. We agree to call these sections the tetrad fields because of the following.

With respect to the holonomic frames $\{\partial_\mu\}$ in the tangent bundle TX , every element $\{H_a\}$ of the linear frame bundle LX takes the form $H_a = H_a^\mu \partial_\mu$. The matrix elements H_a^μ constitute the bundle coordinates on $LX \rightarrow X$ such that the transition functions and the canonical right action of GL_4 on LX take the form

$$H_a'^\mu = \frac{\partial x'^\mu}{\partial x^\lambda} H_a^\lambda, \quad GL_4 \ni g : H_a^\mu \mapsto H_b^\mu g^b{}_a.$$

Let collections of local sections $\tilde{\Psi} = \{z_\epsilon^s\}$ and $\Psi = \{z_\epsilon = \zeta \circ z_\epsilon^s\}$ yield associated atlases of the principal bundles $\widetilde{LX} \rightarrow \Sigma_T$ and $LX \rightarrow \Sigma_T$, respectively. Then the tetrad bundle $\Sigma_T \rightarrow X$ can be provided with bundle coordinates $(x^\lambda, \sigma_a^\mu = H_a^\mu \circ \zeta_\epsilon)$. Components of a tetrad field h with respect to these coordinates are tetrad functions $h_a^\mu = \sigma_a^\mu \circ h$. The familiar relation $g^{\mu\nu} = h_a^\mu h_b^\nu \eta_{ab}$ between tetrad functions and metric functions of the world metric $g : X \xrightarrow{h} \Sigma_T \rightarrow \Sigma$ holds.

There is the well-known topological obstruction to the existence of a Dirac spin structure. Let us restrict our consideration to non-compact manifolds X . Then a Dirac spinor structure exists iff X is parallelizable [22] (e.g., $X = \mathbb{R} \times Z$ because any orientable three-dimensional manifold Z is parallelizable [23]). In this case, the spin frame bundle is unique and all Dirac spin structures are equivalent [22, 24], i.e., all L_s -principal subbundles P^h of \widetilde{LX} are pairwise isomorphic by means of automorphisms of \widetilde{LX} and isomorphic to the trivial L_s -principal bundle $P^0 \rightarrow X$.

Since $\widetilde{LX} \rightarrow \Sigma_T$ is an L_s -principal bundle, one can consider the associated spinor bundle

$$S = (\widetilde{LX} \times V_s)/L_s \rightarrow \Sigma_T \quad (22)$$

over Σ_T [6, 21]. We agree to call it the universal spinor bundle because, given a tetrad field h , the pull-back $S^h = h^* S \rightarrow X$ of S onto X is a spinor bundle on X which is associated to the L_s -principal bundle P^h . Sections of S^h describe Dirac fermion fields in the presence of a tetrad gravitational field h . Given an atlas $\tilde{\Psi}$ of \widetilde{LX} and the corresponding bundle coordinates $(x^\lambda, \sigma_a^\mu)$ on Σ_T , the universal spinor bundle S (22) is endowed with bundle coordinates $(x^\lambda, \sigma_a^\mu, y^A)$, where y^A are coordinates on the spinor space V_s .

The universal spinor bundle $S \rightarrow \Sigma_T$ is a subbundle of the bundle of Clifford algebras which is generated by the bundle of Minkowski spaces

$$E_M = (LX \times M)/L \rightarrow \Sigma_T \quad (23)$$

associated with the L-principal bundle $LX \rightarrow \Sigma_T$. Then due to the representation γ (17), there exists the representation

$$\gamma_\Sigma : T^*X \otimes_{\Sigma_T} S \longrightarrow (\widetilde{LX} \times (M \otimes V))/L_s \rightarrow (\widetilde{LX} \times \gamma(M \otimes V))/L_s = S, \quad (24)$$

defined by the coordinate expression $\gamma_\Sigma(dx^\lambda) = \sigma_a^\lambda \gamma^a$.

Given a world connection K on X , the universal spinor bundle $S \rightarrow \Sigma_T$ is provided with the connection

$$\begin{aligned} A_\Sigma = dx^\lambda \otimes (\partial_\lambda - \frac{1}{4}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b)\sigma_k^\nu K_{\lambda^\mu \nu} L_{ab}{}^A{}_B y^B \partial_A) + \\ d\sigma_k^\mu \otimes (\partial_\mu^k + \frac{1}{4}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b)L_{ab}{}^A{}_B y^B \partial_A). \end{aligned} \quad (25)$$

Given a tetrad field h , its restriction to S^h is the familiar spin connection

$$K_h = dx^\lambda \otimes [\partial_\lambda + \frac{1}{4}(\eta^{kb}h_\mu^a - \eta^{ka}h_\mu^b)(\partial_\lambda h_k^\mu - h_k^\nu K_{\lambda^\mu \nu})L_{ab}{}^A{}_B y^B \partial_A], \quad (26)$$

defined by a world connection K [25, 26]. The connection (25) yields the first order differential operator

$$\begin{aligned} D :: J^1 S \rightarrow T^*X \otimes_{\Sigma_T} S, \\ D = dx^\lambda \otimes [y_\lambda^A - \frac{1}{4}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b)(\sigma_{\lambda k}^\mu - \sigma_k^\nu K_{\lambda^\mu \nu})L_{ab}{}^A{}_B y^B] \partial_A, \end{aligned} \quad (27)$$

on the fiber bundle $S \rightarrow X$. Its restriction to $J^1 S^h \subset J^1 S$ recovers the familiar covariant differential on the spinor bundle $S^h \rightarrow X$ relative to the spin connection (26). Combining (24) and (27) gives the first order differential operator

$$\begin{aligned} \mathcal{D} = \gamma_\Sigma \circ D : J^1 S \rightarrow T^*X \otimes_{\Sigma_T} S \rightarrow S, \\ \mathcal{D} = \sigma_a^\lambda \gamma^{aB}{}_A [y_\lambda^A - \frac{1}{4}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b)(\sigma_{\lambda k}^\mu - \sigma_k^\nu K_{\lambda^\mu \nu})L_{ab}{}^A{}_B y^B], \end{aligned} \quad (28)$$

on the fiber bundle $S \rightarrow X$. Its restriction to $J^1 S^h \subset J^1 S$ is the familiar Dirac operator on the spinor bundle S^h in the presence of a background tetrad field h and a world connection K .

Thus, we come to the model of the metric-affine gravity and Dirac fermion fields. Its configuration space is the bundle product

$$Y = C_K \times_{\Sigma_T} S, \quad (29)$$

coordinated by $(x^\mu, \sigma_a^\mu, k_\mu^\alpha, y^A)$. Let $J_{\Sigma_T}^1 Y$ denote the first order jet manifold of the fiber bundle $Y \rightarrow \Sigma_T$. This fiber bundle can be endowed with the spin connection

$$A_Y = dx^\lambda \otimes (\partial_\lambda - \frac{1}{4}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b)\sigma_k^\nu k_\lambda^\mu{}_\nu L_{ab}{}^A{}_B y^B \partial_A) + \quad (30)$$

$$d\sigma_k^\mu \otimes (\partial_\mu^k + \frac{1}{4}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b)L_{ab}{}^A{}_B y^B \partial_A).$$

With this connection, we obtain the first order differential operator

$$D_Y = dx^\lambda \otimes [y_\lambda^A - \frac{1}{4}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b)(\sigma_{\lambda k}^\mu - \sigma_k^\nu k_\lambda^\mu{}_\nu)L_{ab}{}^A{}_B y^B] \partial_A, \quad (31)$$

and the total Dirac operator

$$\mathcal{D}_Y = \sigma_a^\lambda \gamma^{aB}{}_A [y_\lambda^A - \frac{1}{4}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b)(\sigma_{\lambda k}^\mu - \sigma_k^\nu k_\lambda^\mu{}_\nu)L_{ab}{}^A{}_B y^B], \quad (32)$$

on the fiber bundle $Y \rightarrow X$. Given a world connection $K : X \rightarrow C_K$, the restrictions of the spin connection A_Y (30), the operators D_Y (31) and \mathcal{D}_Y (32) to K^*Y are exactly the spin connection (25), the operators (27) and (28), respectively. The total Lagrangian of the metric-affine gravity and fermion fields is the sum of a metric-affine Lagrangian L_{MA} depending on the variables $\sigma^{\mu\nu} = \sigma_a^\mu \sigma_b^\nu \eta^{ab}$, k_μ^α and the Dirac Lagrangian constructed from the Dirac operator (32) and the spinor metric (19) [6, 21].

Turn now to gauge transformations in this gauge theory. The spin frame bundle $\widetilde{LX} \rightarrow X$ as like as the linear frame one $LX \rightarrow X$ is natural, and it admits the canonical lift of any vector field on X . Hence, the universal spinor bundle $S \rightarrow X$ is also natural, and the canonical lift u_S onto S of vector fields on X exists. The goal is to find its coordinate expression. Difficulties arise because the tetrad coordinates σ_a^μ on Σ_T depend on the choice of an atlas Ψ of the bundle LX . Therefore, non-canonical components appear in the coordinate expression of u_S [21, 26].

Since the fiber bundles $LX \rightarrow X$, $GL_4 \rightarrow GL_4/L$ and, accordingly, the fiber bundles $\widetilde{LX} \rightarrow X$, $\widetilde{GL}_4 \rightarrow L_s$ are trivial, the fiber bundles $\widetilde{LX} \rightarrow \Sigma_T$ and $S \rightarrow \Sigma_T$ are also trivial. Let Ψ be an atlas of $\widetilde{LX} \rightarrow \Sigma_T$ with transition functions constant on fibers of Σ_T . This atlas provides a trivialization $\widetilde{LX} \cong P^0 \times \Sigma_T$ and the corresponding trivialization

$$S \cong S^0 \times \Sigma_T \quad (33)$$

of the universal spinor bundle. With respect to such an atlas, one can define the lift

$$u_S = u^\lambda \partial_\lambda + \partial_\nu u^\alpha \sigma_c^\nu \partial_\alpha^c + \quad (34)$$

$$\frac{1}{4}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b)(\partial_\nu u^\mu \sigma_k^\nu - u^\nu \sigma_{\nu k}^\mu)(-L_{ab}{}^d{}_c \sigma_d^\alpha \partial_\alpha^c + L_{ab}{}^A{}_B y^B \partial_A),$$

$$L_{ab}{}^d{}_c = \eta_{bc} \delta_a^d - \eta_{ac} \delta_b^d,$$

onto S of vector fields $u = u^\lambda \partial_\lambda$ on X . This lift however is not a vector field because it depends on the jet coordinates $\sigma_{\nu k}^\mu$. It is a generalized vector field (see the next Section). Its projection onto S^h in accordance with the splitting (33) is the well-known Lie derivative

$$u_h = u^\lambda \partial_\lambda + \frac{1}{4}(\eta^{kb} h_\mu^a - \eta^{ka} h_\mu^b)(\partial_\nu u^\mu h_k^\nu - u^\nu \partial_\nu h_k^\mu) L_{ab}{}^A{}_B y^B \partial_A \quad (35)$$

of spinor fields in the presence of a tetrad field h [27, 28]. Note that the lift (34) fails to be a Lie algebra homomorphism.

At the same time, one can think of the fiber bundle $S \rightarrow X$ (33) as being associated with the L_s -principal bundle $P^0 \rightarrow X$. Infinitesimal generators of local one-parameter groups of automorphisms of P^0 are represented by sections of the vector bundle $TP^0/L_s \rightarrow X$ which take the coordinate form $u_P = u^\lambda \partial_\lambda + u^m e_m$, where $\{e_m\}$ is the basis for the Lie algebra of L_s . Any vector field u_P yields the vector field

$$u_S = u^\lambda \partial_\lambda + (\partial_\nu u^\alpha \sigma_c^\nu - u^m I_m{}^d{}_c \sigma_d^\alpha) \partial_\alpha^c + u^m I_m{}^A{}_B y^B \partial_A \quad (36)$$

on the associated bundle S . These vector fields can also be regarded as infinitesimal gauge transformations of gauge theory of gravity in the presence of Dirac fermion fields. The total configuration space of this theory is the bundle product

$$Y = S \times_{\Sigma_T} C_K \quad (37)$$

coordinated by $(x^\lambda, \sigma_a^\mu, k_\mu{}^\alpha{}_\beta, y^A)$. The generalized vector fields (34) and the vector fields (36) are extended to the fiber bundle (37) as

$$u_{SK} = u^\lambda \partial_\lambda + \partial_\nu u^\alpha \sigma_c^\nu \partial_\alpha^c + \quad (38)$$

$$\frac{1}{4}(\eta^{kb} \sigma_\mu^a - \eta^{ka} \sigma_\mu^b)(\partial_\nu u^\mu \sigma_k^\nu - u^\nu \sigma_{\nu k}^\mu)(-L_{ab}{}^d{}_c \sigma_d^\alpha \partial_\alpha^c + L_{ab}{}^A{}_B y^B \partial_A) +$$

$$[\partial_\nu u^\alpha k_\mu{}^\nu{}_\beta - \partial_\beta u^\nu k_\mu{}^\alpha{}_\nu - \partial_\mu u^\nu k_\nu{}^\alpha{}_\beta + \partial_{\mu\beta} u^\alpha] \frac{\partial}{\partial k_\mu{}^\alpha{}_\beta},$$

$$u_{SK} = u^\lambda \partial_\lambda + (\partial_\nu u^\alpha \sigma_c^\nu - u^m I_m{}^d{}_c \sigma_d^\alpha) \partial_\alpha^c + u^m I_m{}^A{}_B y^B \partial_A + \quad (39)$$

$$[\partial_\nu u^\alpha k_\mu{}^\nu{}_\beta - \partial_\beta u^\nu k_\mu{}^\alpha{}_\nu - \partial_\mu u^\nu k_\nu{}^\alpha{}_\beta + \partial_{\mu\beta} u^\alpha] \frac{\partial}{\partial k_\mu{}^\alpha{}_\beta}.$$

3 Gauge symmetries

In a general setting, gauge systems on fiber bundle $Y \rightarrow X$ are described as follows [8].

With the inverse system of jet manifolds

$$X \xleftarrow{\pi} Y \xleftarrow{\pi_0^1} J^1 Y \xleftarrow{\quad} \dots J^{r-1} Y \xleftarrow{\pi_{r-1}^r} J^r Y \xleftarrow{\quad} \dots, \quad (40)$$

one has the direct system

$$\mathcal{O}^*X \xrightarrow{\pi^*} \mathcal{O}^*Y \xrightarrow{\pi_0^{1*}} \mathcal{O}_1^*Y \longrightarrow \cdots \mathcal{O}_{r-1}^*Y \xrightarrow{\pi_{r-1}^{r*}} \mathcal{O}_r^*Y \longrightarrow \cdots \quad (41)$$

of graded differential algebras (henceforth GDAs) \mathcal{O}_r^*Y of exterior forms on jet manifolds J^rY with respect to the pull-back monomorphisms π_{r-1}^{r*} . Its direct limit \mathcal{O}_∞^*Y is a GDA consisting of all exterior forms on finite order jet manifolds modulo the pull-back identification.

The projective limit $(J^\infty Y, \pi_r^\infty : J^\infty Y \rightarrow J^r Y)$ of the inverse system (40) is a Fréchet manifold. A bundle atlas $\{(U_Y; x^\lambda, y^i)\}$ of $Y \rightarrow X$ yields the coordinate atlas

$$\{((\pi_0^\infty)^{-1}(U_Y); x^\lambda, y_\Lambda^i)\}, \quad y_{\lambda+\Lambda}^i = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y_\Lambda^i, \quad 0 \leq |\Lambda|, \quad (42)$$

of $J^\infty Y$, where d_μ are the total derivatives (2). Then \mathcal{O}_∞^*Y can be written in a coordinate form where the horizontal one-forms $\{dx^\lambda\}$ and the contact one-forms $\{\theta_\Lambda^i = dy_\Lambda^i - y_{\lambda+\Lambda}^i dx^\lambda\}$ are local generating elements of the $\mathcal{O}_\infty^0 Y$ -algebra \mathcal{O}_∞^*Y . There is the canonical decomposition $\mathcal{O}_\infty^*Y = \oplus \mathcal{O}_\infty^{k,m}Y$ of \mathcal{O}_∞^*Y into $\mathcal{O}_\infty^0 Y$ -modules $\mathcal{O}_\infty^{k,m}Y$ of k -contact and m -horizontal forms together with the corresponding projectors $h_k : \mathcal{O}_\infty^*Y \rightarrow \mathcal{O}_\infty^{k,*}Y$. Accordingly, the exterior differential on \mathcal{O}_∞^*Y is split into the sum $d = d_H + d_V$ of the nilpotent total and vertical differentials

$$d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \quad d_V(\phi) = \theta_\Lambda^i \wedge \partial_i^\Lambda \phi, \quad \phi \in \mathcal{O}_\infty^*Y.$$

Any finite order Lagrangian (1) is an element of $\mathcal{O}_\infty^{0,n}Y$, while

$$\delta L = \mathcal{E}_i \theta^i \wedge \omega = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_\Lambda (\partial_i^\Lambda \mathcal{L}) \theta^i \wedge \omega \in \mathcal{O}_\infty^{1,n}Y \quad (43)$$

is its Euler–Lagrange operator taking values into the vector bundle

$$V^*Y \otimes_Y^n T^*X, \quad (44)$$

where V^*Y denotes the vertical cotangent bundle of $Y \rightarrow X$. Given a Lagrangian L and its Euler-Lagrange operator δL (43), we further abbreviate $A \approx 0$ with an equality which holds on-shell. This means that A is an element of a module over the ideal I_L of the ring $\mathcal{O}_\infty^0 Y$ which is locally generated by the variational derivatives \mathcal{E}_i and their total derivations $d_\Lambda \mathcal{E}_i$. We say that I_L is a differential ideal because, if a local function f belongs to I_L , then every total derivative $d_\Lambda f$ does as well.

A Lagrangian system on a fiber bundle $Y \rightarrow X$ is said to be a gauge theory if its Lagrangian L admits a family of variational symmetries parameterized by elements of a vector bundle $W \rightarrow X$ and its jet manifolds as follows.

Let $\mathfrak{d}\mathcal{O}_\infty^0 Y$ be the $\mathcal{O}_\infty^0 Y$ -module of derivations of the \mathbb{R} -ring $\mathcal{O}_\infty^0 Y$. Any $\vartheta \in \mathfrak{d}\mathcal{O}_\infty^0 Y$ yields the graded derivation (the interior product) $\vartheta \rfloor \phi$ of the GDA $\mathcal{O}_\infty^* Y$ given by the relations

$$\vartheta \rfloor df = \vartheta(f), \quad \vartheta \rfloor (\phi \wedge \sigma) = (\vartheta \rfloor \phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge (\vartheta \rfloor \sigma), \quad f \in \mathcal{O}_\infty^0 Y, \quad \phi, \sigma \in \mathcal{O}_\infty^* Y,$$

and its derivation (the Lie derivative)

$$\mathbf{L}_\vartheta \phi = \vartheta \rfloor d\phi + d(\vartheta \rfloor \phi), \quad \mathbf{L}_\vartheta (\phi \wedge \phi') = \mathbf{L}_\vartheta (\phi) \wedge \phi' + \phi \wedge \mathbf{L}_\vartheta (\phi'), \quad \phi, \phi' \in \mathcal{O}_\infty^* Y. \quad (45)$$

Relative to an atlas (42), a derivation $\vartheta \in \mathfrak{d}\mathcal{O}_\infty^0$ reads

$$\vartheta = \vartheta^\lambda \partial_\lambda + \vartheta^i \partial_i + \sum_{|\Lambda| > 0} \vartheta_\Lambda^i \partial_i^\Lambda, \quad (46)$$

where the tuple of derivations $\{\partial_\lambda, \partial_i^\Lambda\}$ is defined as the dual of that of the exterior forms $\{dx^\lambda, dy_\Lambda^i\}$ with respect to the interior product \rfloor [29].

A derivation ϑ is called contact if the Lie derivative \mathbf{L}_ϑ (45) preserves the contact ideal of the GDA $\mathcal{O}_\infty^* Y$ generated by contact forms. A derivation v (46) is contact iff

$$\vartheta_\Lambda^i = d_\Lambda(\vartheta^i - y_\mu^i \vartheta^\mu) + y_{\mu+\Lambda}^i \vartheta^\mu, \quad 0 < |\Lambda|. \quad (47)$$

Any contact derivation admits the horizontal splitting

$$\vartheta = \vartheta_H + \vartheta_V = \vartheta^\lambda d_\lambda + (v^i \partial_i + \sum_{0 < |\Lambda|} d_\Lambda v^i \partial_i^\Lambda), \quad v^i = \vartheta^i - y_\mu^i \vartheta^\mu. \quad (48)$$

Its vertical part ϑ_V is completely determined by the first summand

$$v = v^i(x^\lambda, y_\Lambda^i) \partial_i, \quad 0 \leq |\Lambda| \leq k. \quad (49)$$

This is a section of the pull-back $VY \times_Y J^k Y \rightarrow J^k Y$, i.e., a k -order VY -valued differential operator on Y . One calls v (49) a generalized vector field on Y .

One can show that the Lie derivative of a Lagrangian L (1) along a contact derivation ϑ (48) fulfills the first variational formula

$$\mathbf{L}_\vartheta L = v \rfloor \delta L + d_H(h_0(\vartheta \rfloor \Xi_L)) + \mathcal{L} d_V(\vartheta_H \rfloor \omega), \quad (50)$$

where Ξ_L is a Lepagean equivalent of L [29]. A contact derivation ϑ (48) is called variational if the Lie derivative (50) is d_H -exact, i.e., $\mathbf{L}_\vartheta L = d_H \sigma$, $\sigma \in \mathcal{O}_\infty^{0, n-1}$. A glance at the expression (50) shows that: (i) ϑ (48) is variational only if it is projected onto X ; (ii) ϑ is variational iff its vertical part ϑ_V is well; (iii) it is variational iff $v \rfloor \delta L$ is d_H -exact. Therefore, we can restrict our consideration to vertical contact derivations $\vartheta = \vartheta_V$. A

generalized vector field v (49) is called a variational symmetry of a Lagrangian L if it generates a variational contact derivation.

In order to define a gauge symmetry [8], let us consider the bundle product E (3) coordinated by (x^λ, y^i, ξ^r) . Given a Lagrangian L on Y , let us consider its pull-back, say again L , onto E . Let ϑ_E be a contact derivation of the \mathbb{R} -ring $\mathcal{O}_\infty^0 E$, whose restriction

$$\vartheta = \vartheta_E|_{\mathcal{O}_\infty^0 Y} = \sum_{0 \leq |\Lambda|} d_\Lambda v^i \partial_i^\Lambda \quad (51)$$

to $\mathcal{O}_\infty^0 Y \subset \mathcal{O}_\infty^0 E$ is linear in coordinates ξ_Σ^r . It is determined by a generalized vector field v_E on E whose projection

$$v : J^k E \xrightarrow{v_E} VE \rightarrow E \times_Y VY$$

is a linear VY -valued differential operator v (4) on E . Let ϑ_E be a variational symmetry of a Lagrangian L on E , i.e.,

$$v_E \rfloor \delta L = v \rfloor \delta L = d_H \sigma. \quad (52)$$

Then one says that v (4) is a gauge symmetry of a Lagrangian L .

In accordance with Noether's second theorem [8], if a Lagrangian L (1) admits a gauge symmetry v (4), its Euler–Lagrange operator (43) obeys the Noether identity

$$\begin{aligned} \Delta \circ \delta L &= 0, \\ \left[\sum_{0 \leq |\Lambda| \leq m} \Delta_r^{i,\Lambda} d_\Lambda \mathcal{E}_i \right] \xi^r \omega &= \left[\sum_{0 \leq |\Lambda| \leq m} (-1)^{|\Lambda|} d_\Lambda (v_r^{i,\Lambda} \mathcal{E}_i) \right] \xi^r \omega = 0, \end{aligned} \quad (53)$$

where

$$\begin{aligned} \Delta &= \xi^r \Delta_r \omega = \sum_{0 \leq |\Lambda| \leq m} \xi^r \Delta_r^{i,\Lambda} (x^\lambda, y_\Sigma^j) \bar{y}_{\Lambda i} \omega = \xi^r \left[\sum_{0 \leq |\Lambda| \leq m} (-1)^{|\Lambda|} d_\Lambda (v_r^{i,\Lambda} \bar{y}_i) \right] \omega, \\ \Delta_r^{i,\Lambda} &= \sum_{0 \leq |\Sigma| \leq m-|\Lambda|} (-1)^{|\Sigma+\Lambda|} C_{|\Sigma+\Lambda|}^{|\Sigma|} d_\Sigma v_r^{i,\Sigma+\Lambda}, \end{aligned} \quad (54)$$

is the Noether operator on the fiber bundle (44), coordinated by $(x^\lambda, y^i, \bar{y}_i)$, with values in the fiber bundle

$$E^* \otimes_Y^n T^* X. \quad (55)$$

For instance, if a gauge symmetry

$$v = (v_r^i \xi^r + v_r^{i,\mu} \xi_\mu^r + v_r^{i,\nu\mu} \xi_{\nu\mu}^r) \partial_i \quad (56)$$

is of second jet order in parameters, the corresponding Noether operator (54) reads

$$\Delta_r^i = v_r^i - d_\mu v_r^{i,\mu} + d_{\nu\mu} v_r^{i,\nu\mu}, \quad \Delta_r^{i,\mu} = -v_r^{i,\mu} + 2d_\nu v_r^{i,\nu\mu}, \quad \Delta_r^{i,\nu\mu} = v_r^{i,\nu\mu}. \quad (57)$$

4 Gauge symmetries in gauge gravitation theories

Turn now to gauge symmetries of gauge gravitation theories mentioned in Section 2.

Let start with the metric-affine gravitation theory on the fiber bundle $\Sigma \times C_K$ with the infinitesimal gauge transformations (12) parameterized by vector fields on X . In this case, the configuration space (3) of the gauge system is the bundle product

$$E = \Sigma \times_X C_K \times_X TX, \quad (58)$$

coordinated by $(x^\lambda, \sigma^{\mu\nu}, k_\lambda^\nu{}_\alpha, \tau^\lambda)$, and the gauge symmetry (4) is the generalized vector field

$$v = (\tau_\nu^\alpha \sigma^{\nu\beta} + \tau_\nu^\beta \sigma^{\alpha\nu} - \tau^\nu \sigma_\nu^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}} + (\tau_\nu^\alpha k_\mu^\nu{}_\beta - \tau_\beta^\nu k_\mu^\alpha{}_\nu - \tau_\mu^\nu k_\nu^\alpha{}_\beta + \tau_{\mu\beta}^\alpha - \tau^\nu k_{\nu\mu}^\alpha{}_\beta) \frac{\partial}{\partial k_\mu^\alpha{}_\beta}. \quad (59)$$

Let us consider the gauge theory of principal connections in the presence of a metric gravitational field on the fiber bundle $C \times \Sigma$ with the infinitesimal gauge transformations (16). The configuration space of the corresponding gauge system is the bundle product

$$E = C \times_X \Sigma \times_X T_G P, \quad (60)$$

coordinated by $(x^\lambda, a_\lambda^r, \sigma^{\alpha\beta}, \tau^\lambda, \xi^r)$, and its gauge symmetry is given by the generalized vector field

$$v = (c_{pq}^r a_\lambda^p \xi^q + \xi_\lambda^r - \tau_\lambda^\mu a_\mu^r - \tau^\mu a_{\mu\lambda}^r) \partial_r^\lambda + (\tau_\nu^\alpha \sigma^{\nu\beta} + \tau_\nu^\beta \sigma^{\alpha\nu} - \tau^\lambda \sigma_\lambda^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}}. \quad (61)$$

The configuration space of the gauge theory of gravity in the presence of Dirac fermion fields is the fiber bundle (37). In the case of infinitesimal gauge transformations (38), the corresponding gauge system is defined on the bundle product

$$E = S \times_{\Sigma_T} C_K \times_{\Sigma_T} TX, \quad (62)$$

coordinated by $(x^\lambda, \sigma_a^\mu, k_\lambda^\nu{}_\alpha, y^A, \tau^\lambda)$. Its gauge symmetry reads

$$v = \frac{1}{2} (\eta^{kd} \eta_{ac} \sigma_\mu^a \sigma_d^\alpha + \delta_\mu^\alpha \delta_c^k) (\tau_\nu^\mu \sigma_k^\nu - \tau^\nu \sigma_{\nu k}^\mu) \partial_\alpha^c + \left[\frac{1}{4} (\eta^{kb} \sigma_\mu^a - \eta^{ka} \sigma_\mu^b) (\tau_\nu^\mu \sigma_k^\nu - \tau^\nu \sigma_{\nu k}^\mu) L_{ab}{}^A{}_B y^B - \tau^\nu y_\nu^A \right] \partial_A + (\tau_\nu^\alpha k_\mu^\nu{}_\beta - \tau_\beta^\nu k_\mu^\alpha{}_\nu - \tau_\mu^\nu k_\nu^\alpha{}_\beta + \tau_{\mu\beta}^\alpha - \tau^\nu k_{\nu\mu}^\alpha{}_\beta) \frac{\partial}{\partial k_\mu^\alpha{}_\beta}. \quad (63)$$

If infinitesimal gauge transformations are vector fields (39), the configuration space of this gauge system is the bundle product

$$E = S \times_{\Sigma_T} C_K \times_{\Sigma_T} T_{L_s} P^0, \quad (64)$$

coordinated by $(x^\lambda, \sigma_a^\mu, k_\lambda^\nu{}_\alpha, y^A, \tau^\lambda, \xi^m)$, and the gauge symmetry is

$$\begin{aligned} v = & (\tau_\nu^\alpha \sigma_c^\nu - \tau^\nu \sigma_{\nu c}^\alpha - \xi^m I_m{}^d{}_c \sigma_d^\alpha) \partial_\alpha^c + (\xi^m I_m{}^A{}_B y^B - \tau^\nu y_\nu^A) \partial_A + \\ & (\tau_\nu^\alpha k_\mu^\nu{}_\beta - \tau_\beta^\nu k_\mu^\alpha{}_\nu - \tau_\mu^\nu k_\nu^\alpha{}_\beta + \tau_{\mu\beta}^\alpha - \tau^\nu k_{\nu\mu}^\alpha{}_\beta) \frac{\partial}{\partial k_\mu^\alpha{}_\beta}. \end{aligned} \quad (65)$$

5 BRST symmetries

In order to introduce BRST symmetries, let us consider Lagrangian systems of even and odd variables. We describe odd variables and their jets on a smooth manifold X as generating elements of the structure ring of a graded manifold whose body is X [8, 29, 30]. This definition reproduces the heuristic notion of jets of ghosts in the field-antifield BRST theory [31, 32].

Recall that any graded manifold (\mathfrak{A}, X) with a body X is isomorphic to the one whose structure sheaf \mathfrak{A}_Q is formed by germs of sections of the exterior product

$$\wedge Q^* = \mathbb{R} \oplus_X Q^* \oplus_X^2 Q^* \oplus_X \cdots, \quad (66)$$

where Q^* is the dual of some real vector bundle $Q \rightarrow X$ of fiber dimension m . In field models, a vector bundle Q is usually given from the beginning. Therefore, we consider graded manifolds (X, \mathfrak{A}_Q) where the above mentioned isomorphism holds, and call (X, \mathfrak{A}_Q) the simple graded manifold constructed from Q . The structure ring \mathcal{A}_Q of sections of \mathfrak{A}_Q consists of sections of the exterior bundle (66) called graded functions. Let $\{c^a\}$ be the fiber basis for $Q^* \rightarrow X$, together with transition functions $c'^a = \rho_b^a c^b$. It is called the local basis for the graded manifold (X, \mathfrak{A}_Q) . With respect to this basis, graded functions read

$$f = \sum_{k=0}^m \frac{1}{k!} f_{a_1 \dots a_k} c^{a_1} \cdots c^{a_k},$$

where $f_{a_1 \dots a_k}$ are local smooth real functions on X .

Given a graded manifold (X, \mathfrak{A}_Q) , let $\mathfrak{d}\mathcal{A}_Q$ be the \mathcal{A}_Q -module of \mathbb{Z}_2 -graded derivations of the \mathbb{Z}_2 -graded ring of \mathcal{A}_Q , i.e.,

$$u(ff') = u(f)f' + (-1)^{[u][f]} fu(f'), \quad u \in \mathfrak{d}\mathcal{A}_Q, \quad f, f' \in \mathcal{A}_Q,$$

where $[.]$ denotes the Grassmann parity. Its elements are called \mathbb{Z}_2 -graded (or, simply, graded) vector fields on (X, \mathfrak{A}_Q) . Due to the canonical splitting $VQ = Q \times Q$, the vertical

tangent bundle $VQ \rightarrow Q$ of $Q \rightarrow X$ can be provided with the fiber bases $\{\partial_a\}$ which is the dual of $\{c^a\}$. Then a graded vector field takes the local form $u = u^\lambda \partial_\lambda + u^a \partial_a$, where u^λ, u^a are local graded functions. It acts on \mathcal{A}_Q by the rule

$$u(f_{a\dots b} c^a \cdots c^b) = u^\lambda \partial_\lambda (f_{a\dots b}) c^a \cdots c^b + u^d f_{a\dots b} \partial_d (c^a \cdots c^b). \quad (67)$$

This rule implies the corresponding transformation law

$$u'^\lambda = u^\lambda, \quad u'^a = \rho_j^a u^j + u^\lambda \partial_\lambda (\rho_j^a) c^j.$$

Then one can show that graded vector fields on a simple graded manifold can be represented by sections of the vector bundle $\mathcal{V}_Q \rightarrow X$, locally isomorphic to $\wedge Q^* \otimes_X (Q \oplus_X TX)$.

Accordingly, graded exterior forms on the graded manifold (X, \mathfrak{A}_Q) are introduced as sections of the exterior bundle $\wedge \mathcal{V}_Q^*$, where $\mathcal{V}_Q^* \rightarrow X$ is the $\wedge Q^*$ -dual of \mathcal{V}_Q . Relative to the dual local bases $\{dx^\lambda\}$ for T^*X and $\{dc^b\}$ for Q^* , graded one-forms read

$$\phi = \phi_\lambda dx^\lambda + \phi_a dc^a, \quad \phi'_a = \rho^{-1b}_a \phi_b, \quad \phi'_\lambda = \phi_\lambda + \rho^{-1b}_a \partial_\lambda (\rho_j^a) \phi_b c^j.$$

The duality morphism is given by the interior product

$$u \rfloor \phi = u^\lambda \phi_\lambda + (-1)^{[\phi_a]} u^a \phi_a.$$

Graded exterior forms constitute the bigraded differential algebra (henceforth BGDA) \mathcal{C}_Q^* with respect to the bigraded exterior product \wedge and the exterior differential d .

Since the jet bundle $J^r Q \rightarrow X$ of a vector bundle $Q \rightarrow X$ is a vector bundle, let us consider the simple graded manifold $(X, \mathfrak{A}_{J^r Q})$ constructed from $J^r Q \rightarrow X$. Its local basis is $\{x^\lambda, c_\Lambda^a\}$, $0 \leq |\Lambda| \leq r$, together with the transition functions

$$c_{\lambda+\Lambda}^a = d_\lambda (\rho_j^a c_\Lambda^j), \quad d_\lambda = \partial_\lambda + \sum_{|\Lambda| < r} c_{\lambda+\Lambda}^a \partial_a^\Lambda, \quad (68)$$

where ∂_a^Λ are the duals of c_Λ^a . Let $\mathcal{C}_{J^r Q}^*$ be the BGDA of graded exterior forms on the graded manifold $(X, \mathfrak{A}_{J^r Q})$. A linear bundle morphism $\pi_{r-1}^r : J^r Q \rightarrow J^{r-1} Q$ yields the corresponding monomorphism of BGDA $\mathcal{C}_{J^{r-1} Q}^* \rightarrow \mathcal{C}_{J^r Q}^*$. Hence, there is the direct system of BGDA

$$\mathcal{C}_Q^* \xrightarrow{\pi_0^{1*}} \mathcal{C}_{J^1 Q}^* \cdots \xrightarrow{\pi_{r-1}^{r*}} \mathcal{C}_{J^r Q}^* \longrightarrow \cdots. \quad (69)$$

Its direct limit $\mathcal{C}_\infty^* Q$ consists of graded exterior forms on graded manifolds $(X, \mathfrak{A}_{J^r Q})$, $r \in \mathbb{N}$, modulo the pull-back identification, and it inherits the BGDA operations intertwined by the monomorphisms π_{r-1}^{r*} . It is a $C^\infty(X)$ -algebra locally generated by the elements $(1, c_\Lambda^a, dx^\lambda, \theta_\Lambda^a = dc_\Lambda^a - c_{\lambda+\Lambda}^a dx^\lambda)$, $0 \leq |\Lambda|$.

In order to regard even and odd dynamic variables on the same footing, let $Y \rightarrow X$ be hereafter an affine bundle (or a subbundle of an affine bundle which need not be affine),

and let $\mathcal{P}_\infty^* Y \subset \mathcal{O}_\infty^* Y$ be the $C^\infty(X)$ -subalgebra of exterior forms whose coefficients are polynomial in the fiber coordinates y_Λ^i on jet bundles $J^r Y \rightarrow X$. Let us consider the product

$$\mathcal{S}_\infty^* = \mathcal{C}_\infty^* Q \wedge \mathcal{P}_\infty^* Y \quad (70)$$

of graded algebras $\mathcal{C}_\infty^* Q$ and $\mathcal{P}_\infty^* Y$ over their common graded subalgebra $\mathcal{O}^* X$ of exterior forms on X [29]. It consists of the elements

$$\sum_i \psi_i \otimes \phi_i, \quad \sum_i \phi_i \otimes \psi_i, \quad \psi \in \mathcal{C}_\infty^* Q, \quad \phi \in \mathcal{P}_\infty^* Y,$$

modulo the commutation relations

$$\begin{aligned} \psi \otimes \phi &= (-1)^{|\psi||\phi|} \phi \otimes \psi, & \psi \in \mathcal{C}_\infty^* Q, & \quad \phi \in \mathcal{P}_\infty^* Y, \\ (\psi \wedge \sigma) \otimes \phi &= \psi \otimes (\sigma \wedge \phi), & \sigma \in \mathcal{O}^* X. \end{aligned} \quad (71)$$

They are endowed with the total form degree $|\psi| + |\phi|$ and the total Grassmann parity $[\psi]$. Their multiplication

$$(\psi \otimes \phi) \wedge (\psi' \otimes \phi') := (-1)^{|\psi'||\phi|} (\psi \wedge \psi') \otimes (\phi \wedge \phi'). \quad (72)$$

obeys the relation

$$\varphi \wedge \varphi' = (-1)^{|\varphi||\varphi'| + [\varphi][\varphi']} \varphi' \wedge \varphi, \quad \varphi, \varphi' \in \mathcal{S}_\infty^*,$$

and makes \mathcal{S}_∞^* (70) into a bigraded $C^\infty(X)$ -algebra. For instance, elements of the ring \mathcal{S}_∞^0 are polynomials of c_Λ^a and y_Λ^i with coefficients in $C^\infty(X)$.

The algebra \mathcal{S}_∞^* is provided with the exterior differential

$$d(\psi \otimes \phi) := (d_{\mathcal{C}} \psi) \otimes \phi + (-1)^{|\psi|} \psi \otimes (d_{\mathcal{P}} \phi), \quad \psi \in \mathcal{C}_\infty^*, \quad \phi \in \mathcal{P}_\infty^*, \quad (73)$$

where $d_{\mathcal{C}}$ and $d_{\mathcal{P}}$ are exterior differentials on the differential algebras $\mathcal{C}_\infty^* Q$ and $\mathcal{P}_\infty^* Y$, respectively. It obeys the relations

$$d(\varphi \wedge \varphi') = d\varphi \wedge \varphi' + (-1)^{|\varphi|} \varphi \wedge d\varphi', \quad \varphi, \varphi' \in \mathcal{S}_\infty^*,$$

and makes \mathcal{S}_∞^* into a BGDA. We agree to call elements of \mathcal{S}_∞^* the graded exterior forms on X . Hereafter, let the collective symbols s_Λ^A stand both for even and odd generating elements c_Λ^a , y_Λ^i of the $C^\infty(X)$ -ring \mathcal{S}_∞^0 . Then the BGDA \mathcal{S}_∞^* is locally generated by $(1, s_\Lambda^A, dx^\lambda, \theta_\Lambda^A = ds_\Lambda^A - s_{\lambda+\Lambda}^A dx^\lambda)$, $|\Lambda| \geq 0$. Since the generating elements s_Λ^A and θ_Λ^A are derived from s^A , the set $\{s^A\}$ is called the local basis for the BGDA \mathcal{S}_∞^* .

Similarly to $\mathcal{O}_\infty^* Y$, the BGDA \mathcal{S}_∞^* is decomposed into \mathcal{S}_∞^0 -modules $\mathcal{S}_\infty^{k,r}$ of k -contact and r -horizontal graded forms together with the corresponding projections h_k and h^r . Accordingly, the exterior differential d (73) on \mathcal{S}_∞^* is split into the sum $d = d_H + d_V$ of the total and vertical differentials

$$d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \quad d_V(\phi) = \theta_\Lambda^A \wedge \partial_A^\Lambda \phi, \quad \phi \in \mathcal{S}_\infty^*.$$

One can think of the elements

$$L = \mathcal{L}\omega \in \mathcal{S}_\infty^{0,n}, \quad \delta(L) = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} \theta^A \wedge d_\Lambda(\partial_A^\Lambda L) \in \mathcal{S}_\infty^{0,n}$$

as being a graded Lagrangian and its Euler–Lagrange operator, respectively.

A graded derivation $\vartheta \in \mathfrak{d}\mathcal{S}_\infty^0$ of the \mathbb{R} -ring \mathcal{S}_∞^0 is said to be contact if the Lie derivative \mathbf{L}_ϑ preserves the ideal of contact graded forms of the BGDA \mathcal{S}_∞^* . With respect to the local basis $(x^\lambda, s_\Lambda^A, dx^\lambda, \theta_\Lambda^A)$ for the BGDA \mathcal{S}_∞^* , any contact graded derivation takes the form

$$\vartheta = \vartheta_H + \vartheta_V = \vartheta^\lambda d_\lambda + (\vartheta^A \partial_A + \sum_{|\Lambda| > 0} d_\Lambda \vartheta^A \partial_A^\Lambda), \quad (74)$$

where $\vartheta^\lambda, \vartheta^A$ are local graded functions [29]. The interior product $\vartheta \rfloor \phi$ and the Lie derivative $\mathbf{L}_\vartheta \phi$, $\phi \in \mathcal{S}_\infty^*$, are defined by the formulae

$$\begin{aligned} \vartheta \rfloor \phi &= \vartheta^\lambda \phi_\lambda + (-1)^{[\phi]_A} \vartheta^A \phi_A, \quad \phi \in \mathcal{S}_\infty^1, \\ \vartheta \rfloor (\phi \wedge \sigma) &= (\vartheta \rfloor \phi) \wedge \sigma + (-1)^{|\phi| + [\phi][\vartheta]} \phi \wedge (\vartheta \rfloor \sigma), \quad \phi, \sigma \in \mathcal{S}_\infty^*, \\ \mathbf{L}_\vartheta \phi &= \vartheta \rfloor d\phi + d(\vartheta \rfloor \phi), \quad \mathbf{L}_\vartheta(\phi \wedge \sigma) = \mathbf{L}_\vartheta(\phi) \wedge \sigma + (-1)^{[\vartheta][\phi]} \phi \wedge \mathbf{L}_\vartheta(\sigma). \end{aligned}$$

The Lie derivative $\mathbf{L}_\vartheta L$ of a Lagrangian L along a contact graded derivation ϑ (74) fulfills the first variational formula

$$\mathbf{L}_\vartheta L = \vartheta_V \rfloor \delta L + d_H(h_0(\vartheta \rfloor \Xi_L)) + d_V(\vartheta_H \rfloor \omega) \mathcal{L}, \quad (75)$$

where Ξ_L is a Lepagean equivalent of a graded Lagrangian L [29].

A contact graded derivation ϑ is said to be variational if the Lie derivative (75) is d_H -exact. A glance at the expression (75) shows that: (i) a contact graded derivation ϑ is variational only if it is projected onto X , and (ii) ϑ is variational iff its vertical part ϑ_V is well. Therefore, we restrict our consideration to vertical contact graded derivations

$$\vartheta = \sum_{0 \leq |\Lambda|} d_\Lambda v^A \partial_A^\Lambda. \quad (76)$$

Such a derivation is completely defined by its first summand

$$v = v^A(x^\lambda, s_\Lambda^A) \partial_A, \quad 0 \leq |\Lambda| \leq k, \quad (77)$$

which is also a graded derivation of \mathcal{S}_∞^0 . It is called the generalized graded vector field. A glance at the first variational formula (75) shows that ϑ (76) is variational iff $v \rfloor \delta L$ is d_H -exact.

A vertical contact graded derivation ϑ (76) is said to be nilpotent if

$$\mathbf{L}_v(\mathbf{L}_v\phi) = \sum_{|\Sigma| \geq 0, |\Lambda| \geq 0} (v_\Sigma^B \partial_B^\Sigma (v_\Lambda^A) \partial_A^\Lambda + (-1)^{[s^B][v^A]} v_\Sigma^B v_\Lambda^A \partial_B^\Sigma \partial_A^\Lambda) \phi = 0 \quad (78)$$

for any horizontal graded form $\phi \in \mathcal{S}_\infty^{0,*}$ or, equivalently, $(\vartheta \circ \vartheta)(f) = 0$ for any graded function $f \in \mathcal{S}_\infty^0$. One can show that ϑ is nilpotent only if it is odd and iff the equality

$$\vartheta(v^A) = \sum_{|\Sigma| \geq 0} v_\Sigma^B \partial_B^\Sigma (v^A) = 0 \quad (79)$$

holds for all v^A [29].

Return now to the original gauge system on a fiber bundle Y with a Lagrangian L (1) and a gauge symmetry v (4). Let us consider the BGDA

$$\mathcal{S}_\infty^*[W; Y] = \mathcal{C}_\infty^* W \wedge \mathcal{P}_\infty^* Y$$

whose local basis is $\{c^r, y^i\}$. Let $L \in \mathcal{O}_\infty^{0,n} Y$ be a polynomial in y_Λ^i , $0 \leq |L|$. Then it is a graded Lagrangian $L \in \mathcal{P}_\infty^{0,n} Y \subset \mathcal{S}_\infty^{0,n}[W; Y]$ in $\mathcal{S}_\infty^*[W; Y]$. Its gauge symmetry v (4) gives rise to the generalized vector field $v_E = v$ on E , and the latter defines the generalized graded vector field v (77) by the formula (5). It is easily justified that the contact graded derivation ϑ (76) generated by v (5) is variational for L . It is odd, but need not be nilpotent. However, one can try to find a nilpotent contact graded derivation (76) generated by some generalized graded vector field (6) which coincides with ϑ on $\mathcal{P}_\infty^* Y$. Then v (6) is called a BRST symmetry.

In this case, the nilpotency conditions (79) read

$$\sum_\Sigma d_\Sigma \left(\sum_{\Xi} v_r^{i,\Xi} c_\Xi^r \right) \sum_\Lambda \partial_i^\Sigma (v_s^{j,\Lambda}) c_\Lambda^s + \sum_\Lambda d_\Lambda (u^r) v_r^{j,\Lambda} = 0, \quad (80)$$

$$\sum_\Lambda \left(\sum_{\Xi} d_\Lambda (v_r^{i,\Xi} c_\Xi^r) \partial_i^\Lambda + d_\Lambda (u^r) \partial_r^\Lambda \right) u^q = 0 \quad (81)$$

for all indices j and q . They are equations for graded functions $u^r \in \mathcal{S}_\infty^0[W; Y]$. Since these functions are polynomials

$$u^r = u_{(0)}^r + \sum_\Gamma u_{(1)p}^{r,\Gamma} c_\Gamma^p + \sum_{\Gamma_1, \Gamma_2} u_{(2)p_1 p_2}^{r, \Gamma_1 \Gamma_2} c_{\Gamma_1}^{p_1} c_{\Gamma_2}^{p_2} + \dots \quad (82)$$

in c_Λ^s , the equations (80) – (81) take the form

$$\sum_\Sigma d_\Sigma \left(\sum_{\Xi} v_r^{i,\Xi} c_\Xi^r \right) \sum_\Lambda \partial_i^\Sigma (v_s^{j,\Lambda}) c_\Lambda^s + \sum_\Lambda d_\Lambda (u_{(2)}^r) v_r^{j,\Lambda} = 0, \quad (83)$$

$$\sum_{\Lambda} d_{\Lambda}(u_{(k \neq 2)}^r) v_r^{j, \Lambda} = 0, \quad (84)$$

$$\sum_{\Lambda} \sum_{\Xi} d_{\Lambda}(v_r^{i, \Xi} c_{\Xi}^r) \partial_i^{\Lambda} u_{(k-1)}^q + \sum_{m+n-1=k} d_{\Lambda}(u_{(m)}^r) \partial_r^{\Lambda} u_{(n)}^q = 0. \quad (85)$$

One can think of the equalities (83) and (85) as being the generalized commutation relations and generalized Jacobi identities of original gauge transformations, respectively [9].

6 BRST symmetries in gauge gravitation theories

Following the procedure in Section 5, let us obtain BRST symmetries of gauge gravitation theories listed in Section 2.

Given a gauge system on the fiber bundle (58), let us consider the BGDA

$$\mathcal{S}_{\infty}^*[TX; \Sigma \times_X C_K] = \mathcal{C}_{\infty}^*[TX] \wedge \mathcal{P}_{\infty}[\Sigma \times_X C_K], \quad (86)$$

whose basis consists of even elements $\sigma^{\alpha\beta}$, $k_{\mu}^{\alpha\beta}$ and odd elements c^{λ} , which are ghosts substituting for parameters τ^{λ} in the gauge symmetry (59). We obtain the nilpotent BRST symmetry

$$\begin{aligned} v = & v^{\alpha\beta} \frac{\partial}{\partial \sigma^{\alpha\beta}} + v_{\mu}^{\alpha\beta} \frac{\partial}{\partial k_{\mu}^{\alpha\beta}} + v^{\lambda} \frac{\partial}{\partial c^{\lambda}} = (\sigma^{\nu\beta} c_{\nu}^{\alpha} + \sigma^{\alpha\nu} c_{\nu}^{\beta} - c^{\lambda} \sigma_{\lambda}^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}} + \\ & (c_{\nu}^{\alpha} k_{\mu}^{\nu\beta} - c_{\beta}^{\nu} k_{\mu}^{\alpha\nu} - c_{\mu}^{\nu} k_{\nu}^{\alpha\beta} + c_{\mu\beta}^{\alpha} - c^{\lambda} k_{\lambda\mu}^{\alpha\beta}) \frac{\partial}{\partial k_{\mu}^{\alpha\beta}} + c_{\mu}^{\lambda} c^{\mu} \frac{\partial}{\partial c^{\lambda}}. \end{aligned} \quad (87)$$

The BGDA of the gauge system on the fiber bundle (60) is $\mathcal{S}_{\infty}^*[T_G P; C \times \Sigma]$ whose odd basis elements c^r and c^{λ} substitute for parameters ξ^r and τ^{λ} in the gauge symmetry (61). The corresponding BRST symmetry reads

$$\begin{aligned} v = & (c_{pq}^r a_{\lambda}^p c^q + c_{\lambda}^r - a_{\mu}^r c_{\lambda}^{\mu} - c^{\mu} a_{\mu\lambda}^r) \frac{\partial}{\partial a_{\lambda}^r} + (\sigma^{\nu\beta} c_{\nu}^{\alpha} + \sigma^{\alpha\nu} c_{\nu}^{\beta} - c^{\lambda} \sigma_{\lambda}^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}} + \\ & (-\frac{1}{2} c_{pq}^r c^p c^q - c^{\mu} c_{\mu}^r) \frac{\partial}{\partial c^r} + c_{\mu}^{\lambda} c^{\mu} \frac{\partial}{\partial c^{\lambda}}. \end{aligned} \quad (88)$$

Turn now to the gauge theory of gravity in the presence of fermion field. Since gauge transformations (35) fail to form a closed algebra, one can not associate to them a nilpotent BRST symmetry. Therefore, let us consider the gauge gravitation system on the fiber bundle (64). Its BGDA is

$$\mathcal{S}_{\infty}^*[T_{L_s} P^0; S \times_{\Sigma_T} C_K],$$

whose odd basis elements c^m and c^λ play the role of ghosts of the gauge symmetry (65). The associated BRST symmetry is

$$\begin{aligned} v = & (c_\nu^\alpha \sigma_c^\nu - c^\nu \sigma_{\nu c}^\alpha - c^m I_m^d{}_c \sigma_d^\alpha) \frac{\partial}{\partial \sigma_c^\alpha} + (c^m I_m^A{}_B y^B - c^\nu y_\nu^A) \partial_A + \\ & (c_\nu^\alpha k_\mu{}^\nu{}_\beta - c_\beta^\nu k_\mu{}^\alpha{}_\nu - c_\mu^\nu k_\nu{}^\alpha{}_\beta + c_{\mu\beta}^\alpha - c^\lambda k_{\lambda\mu}{}^\alpha{}_\beta) \frac{\partial}{\partial k_\mu{}^\alpha{}_\beta} + \\ & \left(-\frac{1}{2} c_{pq}^m c^p c^q - c^\mu c_\mu^m\right) \frac{\partial}{\partial c^m} + c_\mu^\lambda c^\mu \frac{\partial}{\partial c^\lambda}. \end{aligned} \quad (89)$$

7 The BV quantization

As was mentioned above, we restrict our consideration to the BV quantization of the metric-affine gravitation theory. Its original Lagrangian L_{MA} on the fiber bundle $\Sigma \times C_K$ need not be specified, but it possesses the gauge symmetry v (59). It is important that this gauge symmetry is complete and irreducible.

Let us note that any Lagrangian L has gauge symmetries. In particular, there always exist trivial gauge symmetries

$$v = \sum_\Lambda \eta(M)_r^{i,\Lambda} \xi_\Lambda^r, \quad M_r^{i,\Lambda} = \sum_\Sigma T^{i,j,\Lambda,\Sigma} d_\Sigma \mathcal{E}_j, \quad T_r^{j,i,\Lambda,\Sigma} = -T_r^{i,j,\Sigma,\Lambda}.$$

Furthermore, given a gauge symmetry v (4), let h be a linear differential operator on some vector bundle $E' \rightarrow Y$, coordinated by (x^λ, y^i, ξ'^s) , with values in the vector bundle E . Then the composition $v \circ h = v_s^{i,\Lambda} \xi_\Lambda'^s \partial_i$ is a variational symmetry of the pull-back onto E' of a Lagrangian L on Y , i.e., a gauge symmetry of L . In view of this ambiguity, we agree to say that a gauge symmetry v (4) of a Lagrangian L is complete if a different gauge symmetry v' of L factors through v as

$$v'_0 = v \circ h + T, \quad T \approx 0.$$

In the case of a generic metric-affine Lagrangian L_{MA} on the jet manifold $J^1(\Sigma \times C_K)$, the gauge symmetry v (59) is complete. Although there are Lagrangians, e.g., the Hilbert–Einstein scalar curvature and the Yang–Mills type Lagrangian which admit more wide gauge symmetries. Note that a metric-affine Lagrangian L_{MA} usually depends on $k_\mu{}^\alpha{}_\beta$ through the curvature $R_{\lambda\mu}{}^\alpha{}_\beta$ (8). The Euler–Lagrange operator

$$\delta L_{\text{MA}} = \mathcal{E}_{\alpha\beta} d\sigma^{\alpha\beta} \wedge \omega + \mathcal{E}^\mu{}_\alpha{}^\beta dk_\mu{}^\alpha{}_\beta \wedge \omega \quad (90)$$

of a metric-affine Lagrangian takes its values in the fiber bundle

$$(V^* \Sigma \times_X V^* C_K) \bigotimes_{\Sigma \times C_K} \bigwedge^4 T^* X, \quad (91)$$

coordinated by $(x^\lambda, \sigma^{\mu\nu}, k_\lambda^\nu, \bar{\sigma}_{\mu\nu}, \bar{k}_\nu^\lambda)$. Due to the canonical splittings

$$V^*\Sigma = \Sigma \times_X \Sigma^* = \Sigma \times_X \overset{2}{\vee} TX, \quad V^*C_K = C_K \times_X C_K^* = C_K \times_X (\overset{2}{\otimes} TX \otimes T^*X), \quad (92)$$

the fiber bundle (91) is isomorphic to

$$(\Sigma \times_X C_K) \times_X (\Sigma^* \times_X C_K^*) \otimes_X \overset{4}{\wedge} T^*X.$$

A complete gauge symmetry $v \not\approx 0$ (4) is called reducible if there exist a vector bundle $W_0 \rightarrow X$ and a E -valued differential operator $v^0 \not\approx 0$ on the bundle product $E_0 = Y \times W_0$ which is linear on W_0 and obeys the equality $v \circ v^0 \approx 0$. If v^0 is another differential operator possessing these properties, then v^0 factors through v^0 .

The gauge symmetry (59) is irreducible because its term of maximal jet order is $\tau_{\mu\beta}^\alpha \partial_\alpha^{\mu\beta}$. Indeed, let us assume that there exist a vector bundle $W_0 \rightarrow X$ coordinated by (x^λ, w^A) and the above mentioned differential operator v^0 on the bundle product

$$E_0 = \Sigma \times_X C_K \times_X W_0$$

with values in the vector bundle E (58) such that $v \circ v^0 \approx 0$. Given the maximal jet order term

$$\sum_{|\Xi|=k} v_{0A}^{\alpha, \Xi} w_\Xi^A \frac{\partial}{\partial \tau^\alpha} \not\approx 0 \quad (93)$$

of the operator v^0 , the equality $v \circ v^0 \approx 0$ results in the condition

$$\sum_{|\Xi|=k} v_{0A}^{\alpha, \Xi} d_{\mu\beta} w_\Xi^A \frac{\partial}{\partial k_{\mu\alpha}^\beta} \approx 0.$$

It follows that all coefficients $v_{0A}^{\alpha, \Xi}$ and, consequently, the term (93) vanish on-shell.

By virtue of Noether's second theorem, the gauge symmetry (59) defines the Noether operator Δ (54) on the fiber bundle (91) with values in the fiber bundle

$$(\Sigma \times_X C_K \times_X T^*X) \otimes_{\Sigma \times C_K} \overset{4}{\wedge} T^*X, \quad (94)$$

coordinated by $(x^\lambda, \sigma^{\mu\nu}, k_\lambda^\nu, \bar{\tau}_\lambda)$. The formula (57) gives

$$\begin{aligned} \Delta = \tau^\lambda \sum_{|\Lambda|=0,1,2} [\Delta_\lambda^{\alpha\beta, \Lambda} d_\Lambda \bar{\sigma}_{\alpha\beta} + \Delta_{\lambda\mu}{}^\alpha{}_\beta{}^\Lambda d_\Lambda \bar{k}^\mu{}_\alpha{}^\beta] \omega = \\ \tau^\lambda [-(\sigma_\lambda^{\alpha\beta} + 2\sigma_\nu^{\nu\beta} \delta_\lambda^\alpha) \bar{\sigma}_{\alpha\beta} - 2\sigma^{\nu\beta} d_\nu \bar{\sigma}_{\lambda\beta} + (-k_{\lambda\mu}{}^\alpha{}_\beta - k_{\nu\mu}{}^\nu{}_\beta \delta_\lambda^\alpha + k_{\beta\mu}{}^\alpha{}_\lambda + k_{\mu\lambda}{}^\alpha{}_\beta) \bar{k}^\mu{}_\alpha{}^\beta + \\ (-k_{\mu}{}^\nu{}_\beta \delta_\lambda^\alpha + k_{\mu}{}^\alpha{}_\lambda \delta_\beta^\nu + k_\lambda{}^\alpha{}_\beta \delta_\mu^\nu) d_\nu \bar{k}^\mu{}_\alpha{}^\beta + d_{\mu\beta} \bar{k}^\mu{}_\lambda{}^\beta] \omega. \end{aligned} \quad (95)$$

Following the standard BV quantization procedure [4, 5], let us construct the extended Lagrangian which obey certain condition and depends on ghosts and antifields.

In order to introduce antifields of even fields $\sigma^{\alpha\beta}$, $k_\mu^\alpha{}_\beta$ and odd ghosts τ^λ , let us extend the BGDA (86) to the BGDA

$$\mathcal{S}_\infty^*[TX \times_X (\Sigma^* \times_X C_K^*) \otimes_X^4 T^*X; \Sigma \times_X C_K \times_X T^*X \otimes_X^4 T^*X], \quad (96)$$

where the splittings (92) has been used. The local basis for the BGDA (96) is

$$\{\sigma^{\mu\nu}, k_\mu^\alpha{}_\beta, c^\lambda, \bar{\sigma}_{\mu\nu}, \bar{k}^\mu{}_\alpha{}^\beta, \bar{c}_\lambda\}, \quad (97)$$

where the antifields $\bar{\sigma}_{\mu\nu}$, $\bar{k}^\lambda{}_\nu{}^\alpha$ are odd, while the antifields \bar{c}_λ are even. It is convenient to use right graded derivations $\overleftarrow{\partial}$ with respect to antifields. They act on graded functions and forms ϕ on the right by the rule

$$\overleftarrow{\partial}(\phi) = d\phi[\overleftarrow{\partial}] + d(\phi[\overleftarrow{\partial}]), \quad \overleftarrow{\partial}(\phi \wedge \phi') = (-1)^{[\phi]} \overleftarrow{\partial}(\phi) \wedge \phi' + \phi \wedge \overleftarrow{\partial}(\phi').$$

An original metric-affine Lagrangian L_{MA} is assumed to be polynomial in variables $\sigma^{\mu\nu}$, $k_\lambda^\nu{}_\alpha$. Therefore, it is obviously a graded Lagrangian in the BGDA (96). Given the BRST symmetry v (87), let us consider the graded density

$$L = \mathcal{L}\omega = L_{\text{MA}} + v^{\alpha\beta} \bar{\sigma}_{\alpha\beta} \omega + v_\mu^\alpha{}_\beta \bar{k}^\mu{}_\alpha{}^\beta \omega + v^\lambda \bar{c}_\lambda \omega. \quad (98)$$

One can show that it obeys the following conditions

$$v_c^\alpha = \frac{\overleftarrow{\partial} \mathcal{L}}{\partial \bar{\sigma}_\alpha^c}, \quad v_\mu^\alpha{}_\beta = \frac{\overleftarrow{\partial} \mathcal{L}}{\partial \bar{k}^\mu{}_\alpha{}^\beta}, \quad v^\lambda = \frac{\overleftarrow{\partial} \mathcal{L}}{\partial \bar{c}_\lambda}, \quad (99)$$

$$\mathbf{L}_v L = \frac{\overleftarrow{\partial} \mathcal{L}}{\partial \bar{\sigma}_\alpha^c} \frac{\delta L}{\delta \sigma_c^\alpha} + \frac{\overleftarrow{\partial} \mathcal{L}}{\partial \bar{k}^\mu{}_\alpha{}^\beta} \frac{\delta L}{\delta k_{\mu}^\alpha{}_\beta} + \frac{\overleftarrow{\partial} \mathcal{L}}{\partial \bar{c}_\lambda} \frac{\delta L}{\delta c^\lambda} + d_H \sigma = d_H \sigma', \quad (100)$$

$$L|_{\bar{\sigma}=\bar{k}=\bar{c}=0} = L_{\text{MA}}. \quad (101)$$

A glance at these conditions shows that L (98) is a proper solution of the master equation (100) [4, 5]. Thus, it is the desired extended Lagrangian.

One can write different variants of a gauged-fixed Lagrangian (see, e.g., [33] for the problem of the general covariance in quantum gravity). By analogy with the Lorenz gauge in Yang–Mills gauge theory, let us choose the gauge-fixing quantity

$$\mathcal{S}_\beta^\alpha = \sigma^{\lambda\mu} S_{\lambda\mu}{}^\alpha{}_\beta = \sigma^{\lambda\mu} (k_{\lambda\mu}{}^\alpha{}_\beta + k_{\mu\lambda}{}^\alpha{}_\beta), \quad (102)$$

where $S_{\lambda\mu}{}^\alpha{}_\beta$ is given by the expression (9). In order to define a gauge-fixing density, we add even elements B_β^α , odd elements C_α^β and their even antifields \bar{C}_β^α to the local basis for the BGDA (96). Then this gauge-fixing density reads

$$\Phi = C_\alpha^\beta \left(\frac{1}{2} B_\beta^\alpha - \mathcal{S}_\beta^\alpha \right) \omega. \quad (103)$$

The desired gauged-fixed Lagrangian L_{GF} comes from the graded Lagrangian

$$L + B_\beta^\alpha \bar{C}_\alpha^\beta \omega = L_{\text{MA}} + v^{\alpha\beta} \bar{\sigma}_{\alpha\beta} \omega + v_\mu^\alpha \bar{k}^\mu{}_\alpha{}^\beta \omega + v^\lambda \bar{c}_\lambda \omega + B_\beta^\alpha \bar{C}_\alpha^\beta \omega$$

by means of replacement of antifields with the variational derivatives

$$\begin{aligned} \bar{\sigma}_{\alpha\beta} &= \frac{\delta\Phi}{\delta\sigma^{\alpha\beta}} = -C_\nu^\mu (k_{\alpha\beta}{}^\nu{}_\mu + k_{\beta\alpha}{}^\nu{}_\mu), & \bar{k}^\mu{}_\alpha{}^\beta &= \frac{\delta\Phi}{\delta k_\mu^\alpha{}_\beta} = 2d_\lambda (C_\alpha^\beta \sigma^{\lambda\mu}), \\ \bar{c}_\lambda &= \frac{\delta\Phi}{\delta c^\lambda} = 0, & \bar{C}_\alpha^\beta &= \frac{\delta\Phi}{\delta C_\beta^\alpha} = (\frac{1}{2}B_\alpha^\beta - S_\alpha^\beta). \end{aligned}$$

One obtains

$$\begin{aligned} \mathcal{L}_{\text{GF}} &= \mathcal{L}_{\text{MA}} - v^{\alpha\beta} C_\nu^\mu (k_{\alpha\beta}{}^\nu{}_\mu + k_{\beta\alpha}{}^\nu{}_\mu) + 2v^\mu{}_\alpha{}^\beta d_\gamma (C_\alpha^\beta \sigma^{\gamma\mu}) + B_\beta^\alpha (\frac{1}{2}B_\alpha^\beta - S_\alpha^\beta) = \quad (104) \\ &\mathcal{L}_{\text{MA}} - (\sigma^{\lambda\beta} c_\lambda^\alpha + \sigma^{\alpha\lambda} c_\lambda^\beta - c^\lambda \sigma_\lambda^{\alpha\beta}) C_\nu^\mu (k_{\alpha\beta}{}^\nu{}_\mu + k_{\beta\alpha}{}^\nu{}_\mu) + \\ &2(c_\nu^\alpha k_\mu^\nu{}_\beta - c_\beta^\nu k_\mu^\alpha{}_\nu - c_\mu^\nu k_\nu^\alpha{}_\beta + c_\mu^\alpha - c^\lambda k_{\lambda\mu}{}^\alpha{}_\beta) d_\gamma (C_\alpha^\beta \sigma^{\gamma\mu}) + \\ &B_\beta^\alpha (\frac{1}{2}B_\alpha^\beta - \sigma^{\lambda\mu} (k_{\lambda\mu}{}^\beta{}_\alpha + k_{\mu\lambda}{}^\beta{}_\alpha)). \end{aligned}$$

Given the Lagrangian (104), one can not automatically write the generating functional of quantum gravity because the BV quantization procedure fails to provide a functional measure. However, this generating functional at least is a Gaussian integral with respect to variables B_β^α . Therefore, we can replace the Lagrangian (104) with the effective one which takes the familiar form

$$\mathcal{L}_{\text{EGF}} = \mathcal{L}_{\text{MA}} + C_\beta^\alpha \mathcal{M}_{\alpha\lambda}^\beta c^\lambda + \frac{1}{2} \sigma^{\lambda\mu} \sigma^{\gamma\nu} (k_{\lambda\mu}{}^\beta{}_\alpha + k_{\mu\lambda}{}^\beta{}_\alpha) (k_{\gamma\nu}{}^\alpha{}_\beta + k_{\nu\gamma}{}^\alpha{}_\beta),$$

where \mathcal{M} is a linear third order differential operator acting on ghosts c^λ .

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